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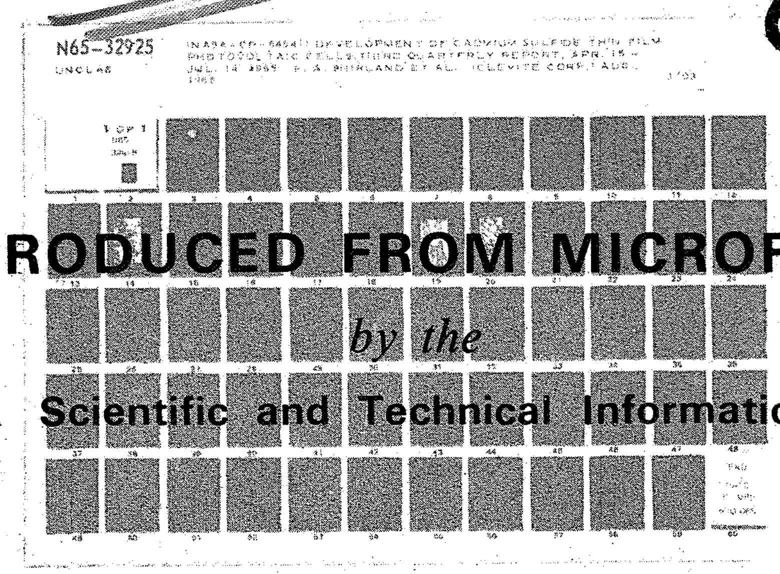

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AN APPLICATION OF THE POWER SERIES METHOD TO SOLUTION OF THE FIRST BOUNDARY VALUE PROBLEM FOR AN EQUATION OF A PARABOLIC TYPE

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An Application of the Power Series Method to Solution of the First Boundary Value Problem for an Equation of a Parabolic Type

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The solution of a non-homogeneous partial differential equation is sought by the power series method developed earlier in [2]. The Laplacian transformation reversion is realized numerically. The calculation technique can be carried out by usual small computers, an arithmometer included, and by electronic computers too.

Let us consider the problem which arises in the study of the process of heat conductivity in the constituent zone of finite length $0 \le x \le b$, with a point source of power A_0 moving in the direction of the axis O_x at a constant velocity u. On the boundary of two zones $x = a$ the contact heat resistance is absent. Let k, ρ, c, κ be the coefficient of heat conductivity, density, specific heat capacity and the coefficient of thermal conductivity. Thermophysical zone properties change continuously.

It is necessary to find the solution of equation

$$\boxed{\rho c \frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) + A_0 \delta (x - ut),\ 0 < x < b,\ t > 0.} \quad (1)$$

This equation satisfies the initial and boundary conditions

$$T(x, 0) = 0,\ T(0, t) = V_0,\ T(b, t) = B_0 \quad (2)$$

and the continuity conditions

$$\boxed{T|_{x=a-0} = T|_{x=a+0};\ k_1 T_x|_{x=a-0} = k_2 T_x|_{x=a+0}} \quad (3)$$

where $\delta(x - ut)$ is the Dirac delta function.

$$\varkappa(x) = \frac{k(x)}{\varrho(x) c(x)};\quad \varkappa(x) = \begin{cases} \varkappa_1(x), & 0 < x < a, \\ \varkappa_2(x), & a < x < b, \end{cases}$$

$$k(x) = \begin{cases} k_1(x), & 0 < x < a, \\ k_2(x), & a < x < b, \end{cases}\quad T(x, t) = \begin{cases} T_1(x, t), & 0 < x < a, \\ T_2(x, t), & a < x < b. \end{cases}$$

The power series method allows us to find the solution of this problem in the class of analytical functions for the case where the coefficients of the equation are analytical functions in its parabolic region. This condition will be required on the basis of the existence theorem. Let us apply the Laplace transformation to the solution of the boundary value problem (1) - (3), from which the following ordinary differential equations are obtained:

$$\frac{1}{k_1}\frac{dk_1}{dx}\frac{d\bar{T}_1(x,p)}{dx}+\frac{d^2\bar{T}_1(x,p)}{dx^2}-\frac{p}{\varkappa_1}\bar{T}_1(x,p)=-\frac{A_0}{k_1}e^{-\frac{px}{v}}, \quad (4)$$

$$\frac{1}{k_2}\frac{dk_2}{dx}\frac{d\bar{T}_2(x,p)}{dx}+\frac{d^2\bar{T}_2(x,p)}{dx^2}-\frac{p}{\varkappa_2}\bar{T}_2(x,p)=-\frac{A_0}{k_2}e^{-\frac{px}{v}} \quad (5)$$

$$a<x<b$$

with the conditions

$$\bar{T}_1(0,p)=\frac{V_0}{p},\ \bar{T}_2(b,p)=\frac{B_0}{p},\ \bar{T}_1(a,p)=\bar{T}_2(a,p),$$
$$k_1(x)\bar{T}_{1,x}|_{x=a}=k_2(x)\bar{T}_{2,x}|_{x=a} \quad (6)$$

The solution to the problem (4) - (6) will be formally sought in the form of the power series

$$\bar{T}_1(x,p)=\sum_{n=0}^{\infty}a_n(x-x_0)^n,\ \bar{T}_2(x,p)=\sum_{n=0}^{\infty}b_n(x-x_0)^n. \quad (7)$$

The functions and their derivatives will also be presented in the form of the power series

$$\bar{T}_{1,x}=\sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n,\ \bar{T}_{2,x}=\sum_{n=0}^{\infty}(n+1)b_{n+1}(x-x_0)^n,$$
$$\bar{T}_{1,xx}=\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n,\ \bar{T}_{2,xx}=\sum_{n=0}^{\infty}(n+1)(n+2)b_{n+2}(x-x_0)^n,$$
$$k_1=\sum_{n=0}^{\infty}\alpha_n(x-x_0)^n,\ k_2=\sum_{n=0}^{\infty}\beta_n(x-x_0)^n,\ k_{1,x}=\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}(x-x_0)^n,$$
$$k_{2,x}=\sum_{n=0}^{\infty}(n+1)\beta_{n+1}(x-x_0)^n,\ \varkappa_1=\sum_{n=0}^{\infty}\gamma_n(x-x_0)^n,\ \varkappa_2=\sum_{n=0}^{\infty}\eta_n(x-x_0)^n.$$

If we let $\frac{1}{k_1(x)}=\sum_{n=0}^{\infty}c_n(x-x_0)^n$, then the method of undetermined coefficients can be used to show that c_n is expressed by the recurrence formula

$$c_0=\frac{1}{\alpha_0},\ c_n=-\frac{1}{\alpha_0}\sum_{k=1}^{n}\alpha_k c_{n-k}\quad (n=1,2,3,\ldots).$$

In the same manner

$$\frac{1}{k_2(x)}=\frac{1}{\beta_0}-\frac{1}{\beta_0}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\beta_k d_{n-k}(x-x_0)^n,$$
$$\frac{1}{\varkappa_1}=\frac{1}{\gamma_0}-\frac{1}{\gamma_0}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\gamma_k f_{n-k}(x-x_0)^n,\ \frac{1}{\varkappa_2}=\frac{1}{\eta_0}-\frac{1}{\eta_0}\sum_{n=1}^{\infty}\sum_{k=1}^{n}\eta_k g_{n-k}(x-x_0)^n,$$

where

$$d_0=\frac{1}{\beta_0},\ d_n=-\frac{1}{\beta_0}\sum_{k=1}^{n}\beta_k d_{n-k};\ f_0=\frac{1}{\gamma_0},\ f_n=-\frac{1}{\gamma_0}\sum_{k=1}^{n}\gamma_k f_{n-k};$$
$$g_0=\frac{1}{\eta_0},\ g_n=-\frac{1}{\eta_0}\sum_{k=1}^{n}\eta_k g_{n-k},\quad (n=1,2,3,\ldots).$$

Using the Cauchy formula [1-2] for the multiplication of power series, we obtain

$$\frac{1}{k_1}\frac{dk_1}{dx}\frac{d\bar{T}_1}{dx}=\frac{1}{\alpha_0}\sum_{n=0}^{\infty}\mu_n^{(2)}(x-x_0)^n,\ \frac{1}{k_2}\frac{dk_2}{dx}\frac{d\bar{T}_2}{dx}=\frac{1}{\beta_0}\sum_{n=0}^{\infty}\lambda_n^{(2)}(x-x_0)^n,$$
$$\frac{1}{\varkappa_1}\bar{T}_1=\frac{1}{\gamma_0}\sum_{n=0}^{\infty}h_n^{(2)}(x-x_0)^n,\ \frac{1}{\varkappa_2}\bar{T}_2=\frac{1}{\eta_0}\sum_{n=0}^{\infty}g_n^{(2)}(x-x_0)^n,$$
$$\frac{1}{k_1}e^{-\frac{px}{v}}=\frac{e^{-\frac{px_0}{v}}}{\alpha_0}\sum_{n=0}^{\infty}\omega^{(n)}(x-x_0)^n,\ \frac{1}{k_2}e^{-\frac{px}{v}}=\frac{e^{-\frac{px_0}{v}}}{\beta_0}\sum_{n=0}^{\infty}\varepsilon_n^{(2)}(x-x_0)^n,$$
$$k_1\bar{T}_{1,x}=\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}l\alpha_l\alpha_{n+1-l}(x-x_0)^n,\ k_2\bar{T}_{2,x}=\sum_{n=0}^{\infty}\sum_{l=1}^{n+1}lb_l\beta_{n+1-l}(x-x_0)^n,$$

where

$$\mu_0^{(1)}=\alpha_1 a_1,\ \mu_n^{(1)}=\alpha_1(n+1)a_{n+1}+\sum_{l=1}^{n}l\alpha_l\mu_{n+1-l}^{(1)},$$
$$\mu_0^{(2)}=c_1,\ \mu_n^{(2)}=(n+1)a_{n+1}-\sum_{l=1}^{n}(n+1-l)a_{n+1-l}\sum_{k=1}^{l}\alpha_k c_{l-k},$$
$$\lambda_0^{(1)}=b_1\beta_1,\ \lambda_n^{(1)}=\beta_1(n+1)b_{n+1}+\sum_{l=1}^{n}lb_l\lambda_{n+1-l}^{(1)}$$

$$\lambda_0^{(2)} = \beta_0,\quad \lambda_n^{(2)} = (n+1)\beta_{n+1} - \sum_{l=1}^{n}(n+1-l)\beta_{n+1-l}\sum_{k=1}^{l}\beta_k d_{l-k};$$

$$h_0^{(2)} = a_0,\quad h_n^{(2)} = a_n - \sum_{l=0}^{n-1} a_l \sum_{k=1}^{n-l}\gamma_k f_{n-l-k};$$

$$g_0^{(2)} = b_0,\quad g_n^{(2)} = b_n - \sum_{l=0}^{n-1} b_l \sum_{k=1}^{n-l}\eta_k e_{n-l-k};$$

$$\omega_0^{(2)} = 1,\quad \omega_n^{(2)} = \frac{(-1)^n p^n}{n!u^n} - \sum_{l=0}^{n-1}\frac{(-1)^l p^l}{l!u^l}\sum_{k=1}^{n-l}\alpha_k c_{n-l-k},$$

$$\tau_0^{(2)} = 1,\quad \tau_n^{(2)} = \frac{(-1)^n p^n}{n!u^n} - \sum_{l=0}^{n-1}\frac{(-1)^l p^l}{l!u^l}\sum_{k=1}^{n-l}\beta_k d_{n-l-k}$$

$$(n = 1, 2, 3, \ldots).$$

Now, by substituting the series obtained into equations (4), (5) and condition (6), by equating the coefficients of equal powers, and after appropriate simplification, a condition which must be satisfied by the coefficients on the boundary of continuity will be obtained, and a_0 and b_0 will be found:

$$a_0 = \frac{V_0}{p},\quad b_0 = \frac{B_0}{p},\quad \begin{cases} a_0 + a_1 a = b_0 + b_1 a(1-\lambda) \\ a_1\alpha_0 = b_1\beta_0 \end{cases},$$

$$a_1 = \frac{\beta_0(B_0 - V_0)}{ap[\beta_0 - \alpha_0(1-\lambda)]},\quad b_1 = \frac{\alpha_0(B_0 - V_0)}{ap[\beta_0 - \alpha_0(1-\lambda)]},$$

$$a_2 = \frac{(1-\lambda)(E_1 - E_2 - E_3) + E_4\alpha_0}{R\alpha_0},\quad b_2 = \frac{E_1 - E_2 - E_3 + E_4\beta_0}{R\beta_0},$$

$$a_n\sum_{l=1}^{n+1} l b_l\beta_{n+1-l} = b_n\sum_{l=1}^{n+1} l a_l\alpha_{n+1-l} \quad (n = 2, 3, \ldots),$$

where

$$R = 2u\gamma_0\eta_0 e^{\frac{ab}{u}}[(1-\lambda)\beta_0 + \alpha_0],$$

$$E_1 = pu\alpha_0\beta_0 e^{\frac{ab}{u}}(a_0\eta_0 + b_0\gamma_0),\quad E_2 = A_0\gamma_0\eta_0(\beta_0 e^{\frac{ab}{u}} + \gamma_0),$$

$$E_3 = u\gamma_0\eta_0 e^{\frac{ab}{u}}(a_1\alpha_1\beta_0 + b_1\alpha_0\beta_1),\quad E_4 = u\gamma_0\eta_0 e^{\frac{ab}{u}}[b_1\beta_1(1-\lambda) - a_1\alpha_1].$$

The coefficients a_n and b_n are determined by the recurrence formulas

$$a_{n+2} = \frac{A_n}{D_n},\quad b_{n+2} = \frac{B_n}{D_n}\quad (n = 1, 2, 3, \ldots),\quad A_n = (n+2)\begin{vmatrix}(n+1)K_n \\ a_{n+1}\beta_0\ L_n\end{vmatrix},$$

$$B_n = (n+2)\begin{vmatrix}K_n(n+1) \\ L_n\ -b_{n+1}\alpha_0\end{vmatrix},\quad D_n = (n+1)(n+2)^2\begin{vmatrix}1 & 1 \\ a_{n+1}\beta_0 & -b_{n+1}\alpha_0\end{vmatrix},$$

$$L_n = b_{n+1}\sum_{l=1}^{n+1} l\alpha_l a_{n+2-l} - a_{n+1}\sum_{l=1}^{n+1} l b_l\beta_{n+2-l}$$

$$K_n = \frac{p}{\gamma_0}\left(a_n - \sum_{l=0}^{n-1} a_l\sum_{k=1}^{n-l}\gamma_k f_{n-l-k}\right) + \frac{p}{\eta_0}\left(b_n - \sum_{l=0}^{n-1} b_l\sum_{k=1}^{n-l}\eta_k e_{n-l-k}\right) -$$

$$-\frac{1}{\alpha_0}\left[\alpha_1(n+1)a_{n+1} + \sum_{l=1}^{n} l a_l\mu_{n+1-l}^{(2)}\right] - \frac{1}{\beta_0}\left[\beta_1(n+1)b_{n+1} +\right.$$

$$\left.+\sum_{l=1}^{n} l b_l\lambda_{n+1-l}^{(2)}\right] - \frac{A_0}{u\alpha_0}\left[\frac{(-1)^n p^n}{n!u^n} - \sum_{l=0}^{n-1}\frac{(-1)^l p^l}{l!u^l}\sum_{k=1}^{n-l}\alpha_k c_{n-l-k}\right] -$$

$$-\frac{A_0 e^{-\frac{ab}{u}}}{u\beta_0}\left[\frac{(-1)^n p^n}{n!u^n} - \sum_{l=0}^{n-1}\frac{(-1)^l p^l}{u^l l!}\sum_{k=1}^{n-l}\beta_k d_{n-k}\right].$$

We now apply the inverse Laplace transformation with the aid of the Legendre polynomials [3]. Let us introduce the transformation

$$\bar{T}(x, p) = \int_0^\infty e^{-pt}T(x, t)\,dt = \int_0^1 U(x, z)z^{p-1}\,dz,$$

where $z = e^{-t}$. Then

$$\bar{T}(x, k+1) = \begin{cases}\sum_{n=0}^{\infty} a_n(k+1)x^n, & 0 < x < a \\ \sum_{n=0}^{\infty} b_n(k+1)(x-b)^n, & a < x < b\end{cases}\quad (k = 0, 1, 2, 3, \ldots).$$

The radius of convergence of series (7) can be expressed numerically to any required accuracy in cases when, starting with some $n \geq N$,

$$\frac{a_n}{a_{n+p}} = \frac{a_{n+p}}{a_{n+2p}} = \ldots = \text{const} = R^p.$$

The solution to equation (1) will be given by the function

$$T(x, t) = U(x, z) = \sum_{k=0}^{\infty}(2k+1)P_k^*(z)r_k^*,$$

where

$$r_k^* = \frac{r_k}{(2k+1)},\quad r_k = \begin{cases}m_k, & 0 < x < a \\ m_k, & a < x < b,\end{cases}$$

$$r_k = (2k+1)\int_0^1 U(x, z)P_k^*(z)\,dz.$$